STRATEGICALLY CHOSEN EXAMPLES LEADING TO PROOF INSIGHT: A CASE STUDY OF A MATHEMATICIAN'S PROVING PROCESS

Elise Lockwood
Oregon State University
elise314@gmail.com

Amy Ellis
University of Wisconsin-Madison
aellis1@wisc.edu

Eric Knuth
University of Wisconsin-Madison
knuth@education.wisc.edu

Muhammed F. Dogan
University of Wisconsin-Madison
mdogan@wisc.edu

Caroline Williams
University of Wisconsin-Madison
caro.williams@gmail.com

Examples play a critical role in the exploration and proving of conjectures. Although proof has been studied extensively, the precise ways in which examples might facilitate successful proofs are not well documented or understood. Working within a larger set of studies that argue for the value of examples in proof-related activity, in this paper we present a case study of one mathematician's work on a conjecture in which his strategic, intentional use of examples led to a proof of that conjecture. By examining his work in detail, we highlight specific mechanisms by which the mathematician's examples led to successful proof production. These mechanisms shed light on precise ways in which examples can directly lead to proof and inform our understanding of the conceptual landscape of the interplay between examples and proof.

Keywords: Advanced Mathematical Thinking, Reasoning and Proof, Post-secondary Education

Introduction

In much of the current literature on teaching proof in school mathematics (e.g., Harel & Sowder, 1998; Stylianides & Stylianides, 2009; Zaslavsky, Nickerson, Stylianides, Kidron, & Winicki, 2012), example-based reasoning is framed as a limited way of thinking that students should avoid. It is true that ways of reasoning about proof that overly rely on examples can be problematic, particularly if examples are perceived by students as valid substitutes for proofs. We posit, however, that these potential limitations of examples have become so prominent in the proof literature that potentially positive roles of examples in proof have not been sufficiently explored. We suggest that the exploration of productive roles of examples in proof offers significant untapped research potential. Given that examples seem to play an essential role in mathematicians' exploration of conjectures and subsequent proof attempts, example-based reasoning strategies can lead to meaningful opportunities for research, illuminating ways in which examples might be used to support students' proof development.

The work presented here builds on recent attempts to learn more about the roles that examples can play in exploring and proving conjectures across a variety of levels (Ellis, et al., 2012). In this paper, we present a case study, examining in detail one mathematician's use of examples in developing a proof of a novel (to him) conjecture. We highlight specific mechanisms through which examples were used as the mathematician formulated a proof. By narrowing our analysis to one mathematician, we can provide significant mathematical detail, allowing for key insights to be shared about how the mathematician leveraged examples in constructing a proof. Through the case study we seek to inform our understanding of the relationship between examples and productive proof-related activity, and we discuss potential implications for the teaching and learning of proof.

Literature Review and Theoretical Framework

Weber (2008) states that, "investigations into the practices of professional mathematicians should have a strong influence on what is taught in mathematics classrooms" (p. 451). A number of other researchers have studied mathematicians' practices under the assumption that their work might provide valuable insight about these practices more generally. For example, Carlson and Bloom (2005) examined mathematicians' problem solving, and Savic (2012) used innovated technology to study what mathematicians did when they reached impasses in their proving processes. In the domain of examples, Epstein and Levy (1995) contend that "Most mathematicians spend a lot of time thinking about and analyzing particular examples," and they go on to note that "It is probably the case that most significant advances in mathematics have arisen from experimentation with examples" (p. 6). Alcock and Inglis (2008) similarly argue that there is often a complex interplay between mathematicians' example-based reasoning activities and their deductive reasoning activities. Several mathematics education researchers have accordingly examined various aspects of the relationship between example-based reasoning and deductive reasoning activities among both mathematicians and mathematics students (e.g., Antonini, 2006; Buchbinder & Zaslavsky, 2009; Ellis, et al., 2012; Iannone, Inglis, Mejia-Ramos, Simpson, & Weber, 2011; Lockwood, et al., 2012, 2013).

The case study presented in this paper builds directly on the findings by Lockwood et al. (2012) in which 219 mathematicians responded to an open ended prompt on a survey: *If you sometimes use examples when exploring a new mathematical conjecture, how do you choose the specific examples you select in order to test or explore the conjecture? What explicit strategies or example characteristics, if any, do you use or consider?* Results of the responses produced a framework for types, uses, and strategies of examples (Lockwood et al., 2012), but here we emphasize one particular use of examples that arose in the mathematicians' responses: proof insight. Fourteen of the mathematicians' responses (5.6%) indicated that they use examples in order to gain insight about how to develop a proof for the conjecture. In addition, in the interview study from which this case study is drawn, there were 23 out of 228 total instances (10.1%) in which mathematicians similarly made statements about examples serving an important role in proving conjectures. While there is evidence that mathematicians might use examples to gain insights into proofs, little is known about the precise mechanisms by which this may happen. We thus build upon such prior work on mathematicians and proving by emphasizing in particular the role of examples as a mathematician produces a proof.

Additionally, the conjecture discussed in this paper is adapted from Alcock & Inglis (2008), in which they discuss different ways in which doctoral students used examples in working with conjectures. Our findings build directly upon their work, as we seek to draw a more in-depth picture of precisely how a mathematician's choice and use of examples helped contribute to his successful proof production.

Methods

The case study presented in this paper is from an interview conducted with a mathematician, Dr. Felton (a pseudonym), as he explored and attempted to prove three mathematical conjectures (due to space, only Conjecture 2 is provided, see Figure 1). Dr. Felton, currently an associate professor in a university mathematics department, received his PhD in mathematics with an emphasis in mathematics education; his area of mathematical expertise is algebra. After working on each conjecture, Dr. Felton was asked clarifying questions about his work. While he was not

given unlimited time (he had approximately 15-20 minutes to explore each conjecture), Dr. Felton was able to sketch of a proof of Conjecture 2 (parts a and b) and Conjecture 3. His work on Conjecture 2 (a task adapted from Alcock & Inglis (2008)) is the subject of this paper.

Conjecture 2
All the numbers below should be assumed to be positive integers.
Definition. An abundant number is an integer *n* whose divisors add up to more than 2*n*.
Definition. A perfect number is an integer *n* whose divisors add up to exactly 2*n*.
Definition. A deficient number is an integer *n* whose divisors add up to less than 2*n*.

Conjecture 2a. A number is abundant if and only if it is a multiple of 6.
Conjecture 2b. If *n* is deficient, then every divisor of *n* is deficient.

Figure 1: Conjecture 2a and 2b

All of the interview problems were chosen because they were a) accessible to the mathematicians (regardless of their area of expertise) but not so clearly obvious that they could be proven immediately, and b) accessible to the interviewer, allowing her to follow the mathematicians' work as well as to ask meaningful follow-up questions. While the choice of conjectures did not exactly simulate a mathematician's personal research, the conjectures enabled us to observe what a mathematician might do as he actually explores and attempts to prove conjectures.

The interview was transcribed, and, having identified Dr. Felton's work on Conjecture 2 as involving an illustrative instance of examples leading to proof development, a member of the research team carefully analyzed Dr. Felton's work on Conjecture 2 in particular. The process involved repeated viewings of his work on this conjecture, identifying and characterizing precise ways in which Dr. Felton used specific examples in develop a proof. Findings were discussed and refined during meetings with other research team members.

Results and Discussion

In this section we present a detailed account of Dr. Felton's work on Conjecture 2, including some analytic discussion; in the subsequent Conclusion section we synthesize the results and highlight salient aspects of his proving process. This section is broken into three subsections: *initial exploration*, *targeted exploration*, and *sketching the proof*.

Initial Exploration

Dr. Felton first solved Conjecture 2a, identifying 6 as a counterexample. He stated that he was not familiar with perfect numbers but had recently thought about 6's property that its proper factors sum to itself, which he learned was the definition of perfect. His choice of counterexample here was relevant, as 6 would be a key aspect of his exploration and proof of Conjecture 2b.

After he initially read Conjecture 2b, his reaction was to think it might be true, because "it's hard for me to imagine a counterexample." Having made this assertion, he began a symbolic proof (in line with what Weber and Alcock (2004) call *syntactic* proof production), writing down expressions such as $a*d=b$ and denoting a as having factors *f1, f2, f3*. However, he did not successfully generate a proof and decided that he would try some examples, saying, "This is going to be an interesting one. I feel like I might want to play with a few examples."

The interviewer asked Dr. Felton both why he had initially attempted a symbolic proof, and why he had shifted from that proof attempt to working with specific examples. He explained these decisions in the excerpt below.

Dr. Felton: Well, probably at first I was, well, partly I started with the algebra because…I was already feeling like it was probably true. And I was hoping maybe I'd get lucky and something would jump out at me, the strategy for proving it…And then the real reason why I went after it with examples, not so much that I thought these would be counterexamples, as I thought they would be good test cases, and they'd maybe give me a feel for how, more information as to maybe why this is true.

Since Dr. Felton suspected the conjecture was true, he had hoped that a strategy for a proof would emerge from the algebraic manipulation. Also, he chose specific examples as test cases, hoping they would inform *why* the conjecture might be true. We address what he meant by "test cases" below.

In the following excerpts we follow Dr. Felton's work as he studied a specific example, 12 (6*2=12). What is noteworthy about this work is the care with which he chose this and subsequent examples, and also how he attended to particular mathematical features of the examples. Specifically, he noted that choosing an example that had a perfect factor (such as 6) was judicious because the factor was barely not deficient.

Dr. Felton: I'm curious about 12, now. So, let's see, 2 times 6 equals 12. We know 6 is perfect, which doesn't necessarily make it deficient. It's not deficient…so actually it's a good choice for a potential counterexample, because it's not deficient, but it's not far from being deficient… So, if something's going to work, my, I guess a good way to do it would be to use a perfect factor.

Interviewer: Okay. As opposed to an abundant one?

Dr. Felton: As opposed to an abundant one…'cause then you're going to get tons of stuff which is going to make it harder for you to not make up the difference with the extra.

He would later go on to say that by choosing an example with a factor that was perfect (as opposed to abundant), he was targeting a "test case," or a "boundary case." He described the perfect number as being a boundary situation, "I mean, it's definitely the boundary case here 'cause you have two definitions and perfect is in the middle of the two of them." He later said, "I mean, my emphasis, my focus will probably always be on a boundary case in that kind of a situation… Like what's that critical point when you switch from being one to the other." These comments exemplify Dr. Felton's strategic choice of examples in helping him understand the conjecture. Indeed, Dr. Felton situated the examples he chose within the larger picture of what he was trying to accomplish. We have evidence that by choosing 12 (which has a perfect factor) he was not actually trying to locate a counterexample, but rather to gain insight into the conjecture: "Yeah, I'm mostly, just to kind of play with it. Like, I don't, I really have no belief that there could be a counterexample. But I'm pretty sure this one is true." He acknowledged that he likely would not find a counterexample, and he recognized that while he might not break the conjecture, he could potentially gain insight into the situation, and possibly formulate a proof, by exploring the boundary case 6*2=12.

In working through this example, Dr. Felton wrote out the factors of 12. Below, we see that he made an important observation about this example, noting a specific property, namely that his factor of choice, 6, was exactly half of 12.

Dr. Felton: So 12 has 1, 2, 3, 4, 6. And it also includes 12 in this calculation. Okay, so, so we're looking at this going 1, 2, 3, 6. Um, those guys are giving me the 12, oh, wait a minute. Hold your horses, here. Okay, is that just 'cause it's a half, though?…You're already getting half way there with that one [the 6]. This isn't going to be very helpful when we do, there's

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no way this is going to be deficient… because there's already a 12 in that one. This, this conjecture is feeling quite true.

Dr. Felton suspected that the conjecture was true, and his work with the example 12 seemed to have confirmed that, but he became aware of this special property that he identified, and he was cognizant that he perhaps had come upon a special case. We interpret that he noted that for his example *e*, if a factor *f* is perfect and is half of *e*, then because it is perfect all of its factors will sum to *f*. But *f* is also already somewhere in the list of *e*'s factors, and this guarantees that the factors of *e* will already sum to at least twice *f* (which is *e*), making *e* not deficient. This realization that he had chosen an example with a perfect factor that was exactly half of the number led him to another strategic example choice, and he noted "I'm just going to kind of play with one more, see if it's a generalizable phenomenon."

Targeted Exploration

We call this section *targeted exploration*, as his observation about 6*2=12 resulted in intentional examples choices that followed a particular line of inquiry. Based on properties he identified about his initial example described above, he chose 18 (6*3=18) and again paid attention to the factor that was half of his example, in this case 9. Interestingly, in the excerpt below, Dr. Felton made a technical error in his reasoning, because 9 is actually deficient (its proper factors sum to 4, which is less than 9). However, his observation in the excerpt is still insightful, and it played an important role in his subsequent example choice. The point to glean from the following exchange is his observation that if half a number is already not deficient, then the original number will not be deficient.

Dr. Felton: Because as long, it's, if it's [the factor that is half of the original number] not deficient, that guarantees that your number is not deficient, so it's like a contrapositive thing going on. Right, so nine is not deficient, it's going to have, the sum of its factors is going to give you two times nine.

Interviewer: Oh, I see. Yeah. So you're saying it's an even number and half that number is already not deficient…

Dr. Felton: Right. Then the original won't be deficient.

In spite of his error, this was a key observation, and it led him to choose an example with a particular property. The excerpt below shows that his work with 12 and 18 made him want to choose an example, "something like 6 times 11. I want to have a big, like, not much stuff between 6 and the whole number." We see, then, that his initial choice was based on 6 being perfect, and he chose 6*2=12 as a test, or boundary, case. Then, in examining that example he noticed the special property that arose because 6 was half of 12. This led him intentionally to choose an example in which 6 was not half of the original number, and he selected 6*3=18 to see if his observation is a "generalizable phenomenon." He similarly scrutinized this example, and the way in which 9 interacted with 18 made him move on to select an example that did not have "much stuff between 6 and the whole number." Based on this criterion, he chose 6*11=66, and this ended up being the example that would provide the key insight for the proof.

Having justified his next selection of 6*11=66, Dr. Felton then spent time investigating this chosen example, first writing out factors of 6*11=66. Then, as he studied the factors of 66 (see Figure 2), he made the following observation. As his last sentence suggests, his observation about the "duplication of the perfectness of 6" was enough to make him think that he could come up with a proof.

Dr. Felton: It's almost like you get, like a duplication of the perfect-ness of 6 that shows up in this piece here.

Interviewer: Okay, how so?
Dr. Felton: So, so, like this 1, 2, 3 adds up to 6…11, 22, 33 actually adds up to 66. So I'm feeling like I probably ought to be able to prove that this is a true statement.

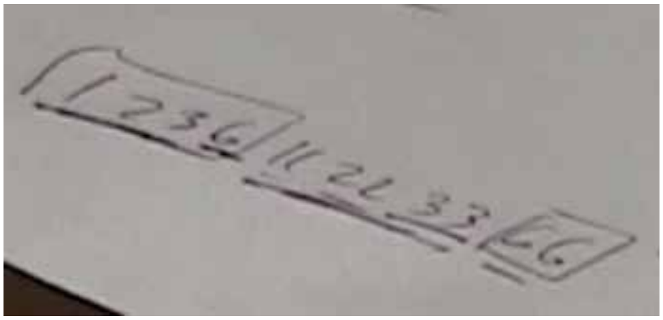

Figure 2: The Perfect-ness of 6 in the Factors of 66

Sketching the Proof

As alluded to previously, Dr. Felton was trying to prove the contrapositive of Conjecture 2b. For the sake of space we briefly summarize his proof, but Figure 3 shows him writing out the sketch of proof and simultaneously referring back specifically to his example in doing so. (In his notation, *b* is the number chosen as the example (*b*=66), *a* is the factor of interest (*a*=6), and *d* is another divisor of *b* (*d*=11)). He was able to prove the contrapositive, arguing that that if a factor of *b*, *a*, is not deficient, then it has factors *f1* through *fk* whose sum is greater than *a.* Then *d*f1* through *d*fk* must also be factors of *b* that are distinct from those factors of *a* (and since *f1* through *fk* are strictly less than *a, d*f1* through *d*fk* must be distinct from *b*). He noted that the sum of *d*f1* through *d*fk* must be greater than or equal to *d*a*, which is itself already a copy of *b*. The sum of *b*'s factors, then, includes *d*f1* through *d*fk,* which is greater or equal to *d*a*, and *b* itself. This is greater than or equal to two copies of *b*, and thus by the definition of deficient, *b* itself cannot be deficient.


Figure 3: Referring to the Example in Writing the Proof

What is most interesting to us is not that he proved the conjecture, but rather the precise role that his example 6*11=66 played in his development of this proof. In reflecting on his proof, he made several statements that highlighted the importance of the example. Specifically, the nature of the multiplication by 11 allowed him to see that certain factors (the multiples of 6) would show up in the complete list of factors. While this is a property that he asserted, "is clearly always going to work out," he acknowledged that the nature of the number 11 made that particularly salient for him. This focus on structure is seen in the following excerpt.

Interviewer: You said something about, maybe, start, being able to start a proof based on that observation you found in that six times eleven…
Dr. Felton: Right, which is kind of what I was thinking of was that the perfect-ness of the, um, 6 is basically copied, replicated by these multiples. Each multiple of the factor of 6. So I've got my 1, 2, 3, 6 here. Right now the 1, 2, 3, adds up to 6. And then multiply each of those by 1, those are also in my list of factors, and I add those up, I get 66. Which, kind of is clearly always going work out…But then when I wrote it down it was actually quite helpful because, I mean, I was also benefited by the choice of 11… Because you multiply by 11, it looks very much like the number you started with before you multiplied by 11… And so, it was much more transparent that the structure on this 11, 22, 33, 66 mirrored the 1, 2, 3, 6.

Conclusion and Implications

In this section we highlight three aspects of Dr. Felton's use of examples as he developed his proof of Conjecture 2b, and we suggest that these specify particular ways in which examples can be leveraged in the proving process. First, Dr. Felton demonstrated that he chose examples in terms of the larger task at hand. He explored a boundary case as a potential counterexample not to attempt to disprove the conjecture, but rather to gain insight into the situation. This activity suggests that he had an understanding of the logical structure of the conjecture, and by looking at an example that would not satisfy the conclusion he was setting himself up for a proof of the contrapositive.

Second, Dr. Felton drew upon particular mathematical properties of his examples, and these led him to strategic subsequent example choices. As described above, the sequence of examples he chose were based on careful mathematical analyses of each example. This activity suggests that Dr. Felton did not choose his examples thoughtlessly, nor did he use his examples merely to check whether the conjecture might be true. Rather he intentionally chose examples, carefully examined their mathematical properties, and used mathematical insight to understand the situation and inform ensuing example selection.

Third, Dr. Felton leveraged the structure of one insightful example in formulating a proof of the conjecture. The nature of the number 11, and the way in which it multiplies, allowed him to recognize in the particular example a key insight about the proof – namely that for all the factors $f1$ through fk of a, $d*f1$ through $d*fk$ would also be factors of b (for $a*b = b$). While he granted that the choice of 11 was fortuitous, he capitalized on this important structure and was able to translate it into the more general argument. As we see in Figure 3 above, he specifically aligned his writing of the proof with the structure of that particular example. This is an instance in which an informed example choice directly affected proof development.

We have previously argued for the value of strategic example choice and progressions of examples, and we have made the case that example structure can play a part in deductive reasoning (Ellis et al., 2012). Additionally, mathematicians have indicated that examples can shed light on proof or lead to proof insight (Lockwood, et al., 2012), but there is not much information on how this occurs. Here, however, we have evidence for how such example-related activity led to a proof in a precise way. Dr. Felton made informed decisions in initial example choice, and he targeted examples with certain mathematical properties. He carefully studied the examples and drew out salient insights from each of them. We do not claim that this is how every conjecture is proved, nor is it how Dr. Felton would even prove every conjecture, nor is it necessarily how people always *ought* to prove conjectures. However, this case study contributes to our understanding of the conceptual landscape of the interplay between examples and proof

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production, pointing to some mechanisms by which example-related activity might lead to proof. Dr. Felton's work also offers some potential pedagogical insight. Students might benefit from modeling his careful selection of a progressive sequence of examples, or from leveraging the structure of a specific example, as he did. Additionally, Dr. Felton exhibited particular attitudes toward examples that could benefit students – he was willing to explore with boundary cases, was cautious of overreliance on a special case, and always seemed to keep the larger proving process in mind. Such facility with and attitudes toward examples might provide starting points for meaningful pedagogical interventions that positively frame examples in the context of proof.

Acknowledgments

The authors wish to thank the other members of the IDIOM Project team. The research is supported by the National Science Foundation (NSF) under grant DRL-0814710. The opinions herein are those of the authors and do not necessarily reflect the views of the (NSF).

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